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STABILITY OF A MULTI-RATE METHOD FOR NUMERICAL INTEGRATION OF ODE'S

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Abstract—This paper derives sufficient conditions for the absolute stability of a certain multi-rate method of numerical integration of systems of first-order ODE's which have been separated into two subsystems, the second system being made up of the faster-response equation. It is assumed that the subsystems are integrated with fourth-order and third-order Runge-Kutta methods, respectively. It is shown that there will be regions of stability, provided the original system is sufficiently diagonally dominant. If the subsystems are weakly coupled, the regions of stability are nearly as large as the classical regions.

1. INTRODUCTION

Consider the following nonlinear initial-value problem:

$$\mathbf{x}' = \mathbf{q}(t, x), \quad \mathbf{x}(t_{i-1}) = \mathbf{x}_{i-1},$$

where $\mathbf{x}$ and $\mathbf{q}$ are vectors. We will write the system in the form

$$\begin{aligned}\mathbf{y}' &= \mathbf{f}(t, \mathbf{y}, \mathbf{z}), \quad \mathbf{y}(t_{i-1}) = \mathbf{y}_{i-1},\\ \mathbf{z}' &= \mathbf{g}(t, \mathbf{y}, \mathbf{z}), \quad \mathbf{z}(t_{i-1}) = \mathbf{z}_{i-1},\end{aligned}$$

where the second subsystem consists of the equations with the faster responses. It is shown in [1,2] that the solution to the original problem is approximately that of the solution obtained by solving the initial value problem

$$\mathbf{y}' = \mathbf{f}[t, \mathbf{y}, \mathbf{z}^*(t, \mathbf{y})], \quad \mathbf{y}(t_{i-1}) = \mathbf{y}_{i-1},$$

where $\mathbf{z}^*(t, \mathbf{y})$ is the solution to the initial value problem

$$\frac{d\mathbf{z}}{d\tau} = \mathbf{g}[\tau, \mathbf{h}(\tau, t, \mathbf{y}), \mathbf{z}], \quad \mathbf{z}(t_{i-1}) = \mathbf{z}_{i-1},$$

integrated from $\tau = t_{i-1}$ to $\tau = t$ with

$$\mathbf{h}(\tau, t, \mathbf{y}) = \mathbf{y}_{i-1} + (\tau - t_{i-1})\,\mathbf{y}'_{i-1} + \left(\frac{\tau - t_{i-1}}{t - t_{i-1}}\right)^2 [\mathbf{y} - \mathbf{y}_{i-1} - (t - t_{i-1})\,\mathbf{y}'_{i-1}],$$

with $\mathbf{y}'_{i-1} = \mathbf{f}(t_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1})$ and with t and $\mathbf{y}$ being regarded as parameters. It was shown in [1,2] that the separation produces errors in the $\mathbf{y}$ and $\mathbf{z}$ solutions which are $O(\Delta t_i^5)$ and $O(\Delta t_i^4)$, respectively, when the equations are integrated from $t = t_{i-1}$ to $t = t_i = t_{i-1} + \Delta t_i$.

We will be concerned with the stability of the method which consists of integrating the $\mathbf{y}' = \mathbf{f}$ and $\mathbf{z}' = \mathbf{g}$ subsystems with fourth-order and third-order Runge-Kutta methods, respectively.

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2. STABILITY ANALYSIS

Consider the initial value problems

$$\mathbf{y}' = \mathbf{F}(t, \mathbf{y}), \quad \mathbf{y}(t_{i-1}) = \mathbf{y}_{i-1}, \tag{1}$$

on the intervals $[t_{i-1}, t_i]$, where $i = 1, 2, \dots,$

$$\mathbf{F}(t, \mathbf{y}) = A_{11}\,\mathbf{y} + A_{12}\,\mathbf{z}^*(t, \mathbf{y}),$$

and $\mathbf{z}^*(t, \mathbf{y})$ is the solution at $\tau = t$ to the initial value problem

$$\frac{d\mathbf{z}}{d\tau} = \mathbf{G}(\tau, \mathbf{z}; t, \mathbf{y}), \quad \mathbf{z}(t_{i-1}) = \mathbf{z}_{i-1}, \tag{2}$$

integrated from $\tau = t_{i-1}$ to $\tau = t$ with

$$G(\tau, \mathbf{z}; t, \mathbf{y}) = A_{21}\left\{\mathbf{y}_{i-1} + (\tau - t_{i-1})\mathbf{y}'_{i-1} + \left(\frac{\tau - t_{i-1}}{t - t_{i-1}}\right)^2 [\mathbf{y} - \mathbf{y}_{i-1} - (t - t_{i-1})\mathbf{y}'_{i-1}]\right\} + A_{22}\,\mathbf{z},$$

and with t and $\mathbf{y}$ being regarded as parameters. Here, $\mathbf{y}'_{i-1} = A_{11}\,\mathbf{y}_{i-1} + A_{12}\,\mathbf{z}_{i-1}$, and it is understood that $\mathbf{y}_i = \mathbf{y}(t_i)$ and $\mathbf{z}_i = \mathbf{z}(t_i) = \mathbf{z}^*(t_i, \mathbf{y}(t_i))$.

It will be assumed that equation (1) will be integrated with any fourth-order Runge-Kutta method. The formula has the form

$$\mathbf{y}_i = \mathbf{y}_{i-1} + \sum_{q=1}^{4} \alpha_q\,\mathbf{k}_q, \tag{3}$$

where

$$\mathbf{k}_q = \Delta t\,\mathbf{F}\left(t_{i-1} + \delta_{q-1}\,\Delta t,\ \mathbf{y}_{i-1} + \sum_{\mu=1}^{3} \delta_{q-1,\mu}\,\mathbf{k}_\mu\right),$$

for $q = 1, 2, 3, 4$. Here, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$, $\delta_{0\mu} = \delta_0 = 0$, $\delta_{11} = \delta_1$, $\delta_{12} = \delta_{13} = 0$, $\delta_{21} + \delta_{22} = \delta_2$, $\delta_{23} = 0$, and $\delta_{31} + \delta_{32} + \delta_{33} = \delta_3$.

It will be supposed that equation (2) will be integrated with any third-order Runge-Kutta method. It has the form

$$\mathbf{z}_{ki} = \mathbf{z}_{k-1,i} + \sum_{p=1}^{3} \beta_p\,\widetilde{\mathbf{k}}_p, \tag{4}$$

where

$$\widetilde{\mathbf{k}}_p = \Delta\tau\,\mathbf{G}\left[t_{i-1} + (k-1)\,\Delta\tau + \gamma_{p-1}\,\Delta\tau,\ \mathbf{z}_{k-1,i} + \sum_{\mu=1}^{2} \gamma_{p-1,\mu}\,\widetilde{\mathbf{k}}_\mu;\ t, \mathbf{y}\right],$$

for $p = 1, 2, 3$. Here, $\beta_1 + \beta_2 + \beta_3 = 1$, $\gamma_{0\mu} = \gamma_0 = 0$, $\gamma_{11} = \gamma_1$, $\gamma_{12} = 0$, and $\gamma_{21} + \gamma_{22} = \gamma_2$. We take $\Delta\tau = \Delta t/N$ where N is a positive integer. Equation (2) is to be integrated over the intervals $[t_{i-1}, t_{i-1} + \delta_j\,\Delta t]$, where $j = 1, 2, 3, 4$ and $\delta_4 = 1$. We assume that $\delta_j N$ is an integer. The number of steps in the four intervals will be $\delta_1 N$, $\delta_2 N$, $\delta_3 N$, and N, respectively. Steps will be over the intervals given by $t_{i-1} + (k-1)\,\Delta\tau \le \tau \le t_{i-1} + k\,\Delta\tau$, $k = 1, 2, \dots, \delta_j N$. We take $\mathbf{z}_i = \mathbf{z}_{\delta_j N, i-1}$. We see that

$$\begin{aligned}\mathbf{k}_i &= \Delta t\,\mathbf{F}(t_{i-1}, \mathbf{y}_{i-1}) = \Delta t\,A_{11}\,\mathbf{y}_{i-1} + \Delta t\,A_{12}\,\mathbf{z}^*(t_{i-1}, \mathbf{y}_{i-1}) \\ &= \Delta t\,A_{11}\,\mathbf{y}_{i-1} + \Delta t\,A_{12}\,\mathbf{z}_{i-1}.\end{aligned}$$

Similarly, for $j = 1, 2, 3, 4$, we have

$$\begin{aligned}\mathbf{k}_j &= \Delta t\,A_{11}\,\mathbf{y}_{i-1} + \sum_{\mu=1}^{3} \delta_{j-1,\mu}\,\Delta t\,A_{11}\,\mathbf{k}_\mu \\ &\quad + \Delta t\,A_{12}\,\mathbf{z}^*\left(t_{i-1} + \delta_{j-1}\,\Delta t,\ \mathbf{y}_{i-1} + \sum_{\mu=1}^{3} \delta_{j-1,\mu}\,\mathbf{k}_\mu\right).\end{aligned} \tag{5}$$

Next, we calculate $\mathbf{z}^*\left(t_{i-1}+\delta_{j-1}\,\Delta t,\ \mathbf{y}_{i-1}+\sum_{\mu=1}^{4}\delta_{j-1,\mu}\,\mathbf{k}_\mu\right)$ for $j=2,3,4,5$, where $\delta_4=1$, $\delta_{4\mu}=\alpha_\mu$, and $\delta_{j-1,4}=0$. Thus,

$$
\begin{aligned}
\widetilde{\mathbf{k}}_p &= \Delta\tau\,\mathbf{G}\Bigg[t_{i-1}+(k-1)\,\Delta\tau+\gamma_{p-1}\,\Delta\tau,\ \mathbf{z}_{k-1,i}+\sum_{\mu=1}^{2}\gamma_{p-1,\mu}\,\widetilde{\mathbf{k}}_\mu;\\
&\qquad t_{i-1}+\delta_{j-1}\,\Delta t,\ \mathbf{y}_{i-1}+\sum_{\mu=1}^{4}\delta_{j-1,\mu}\,\mathbf{k}_\mu\Bigg]\\
&= \Delta\tau\,A_{21}\Bigg[\mathbf{y}_{i-1}+(k-1+\gamma_{p-1})\,\Delta\tau\,\mathbf{y}'_{i-1}\\
&\quad+\left(\frac{(k-1+\gamma_{p-1})\,\Delta\tau}{\delta_{j-1}\,\Delta t}\right)^2\left(\sum_{\mu=1}^{4}\delta_{j-1,\mu}\,\mathbf{k}_\mu-\delta_{j-1}\,\Delta t\,\mathbf{y}'_{i-1}\right)\Bigg]\\
&\quad+\Delta\tau\,A_{22}\,\mathbf{z}_{k-1,i}+\Delta\tau\sum_{\mu=1}^{2}\gamma_{p-1,\mu}\,A_{22}\,\widetilde{\mathbf{k}}_\mu,\\
\widetilde{\mathbf{k}}_1 &= (\Delta\tau\,A_{21})\,\mathbf{y}_{i-1}+(k-1)(\Delta\tau\,A_{21})\,\Delta\tau\,\mathbf{y}'_{i-1}\\
&\quad+\left(\frac{k-1}{\delta_{j-1}N}\right)^2(\Delta\tau\,A_{21})\left(\sum_{\mu=1}^{4}\delta_{j-1,\mu}\,\mathbf{k}_\mu-\delta_{j-1}\,\Delta t\,\mathbf{y}'_{i-1}\right)+(\Delta\tau\,A_{22})\mathbf{z}_{k-1,i},\\
\widetilde{\mathbf{k}}_2 &= (\Delta\tau\,A_{21})\,\mathbf{y}_{i-1}+(k-1+\gamma_1)(\Delta\tau\,A_{21})\,\Delta\tau\,\mathbf{y}'_{i-1}\\
&\quad+\left(\frac{k-1+\gamma_1}{\delta_{j-1}N}\right)^2(\Delta\tau\,A_{21})\left(\sum_{\mu=1}^{4}\delta_{j-1,\mu}\,\mathbf{k}_\mu-\delta_{j-1}\,\Delta t\,\mathbf{y}'_{i-1}\right)\\
&\quad+\gamma_{11}(\Delta\tau\,A_{22})\,\widetilde{\mathbf{k}}_1+(\Delta\tau\,A_{22})\,\mathbf{z}_{k-1,i}.
\end{aligned}
$$

Substituting for $\widetilde{\mathbf{k}}_1$, we obtain

$$
\begin{aligned}
\widetilde{\mathbf{k}}_2 &= [I+\gamma_{11}(\Delta\tau\,A_{22})](\Delta\tau\,A_{21})\,\mathbf{y}_{i-1}+[(k-1+\gamma_1)\,I+\gamma_{11}(k-1)(\Delta\tau\,A_{22})](\Delta\tau\,A_{21})\,\Delta\tau\,\mathbf{y}'_{i-1}\\
&\quad+\left[\left(\frac{k-1+\gamma_1}{\delta_{j-1}N}\right)^2 I+\gamma_{11}\left(\frac{k-1}{\delta_{j-1}N}\right)^2(\Delta\tau\,A_{22})\right](\Delta\tau\,A_{21})\left(\sum_{\mu=1}^{4}\delta_{j-1,\mu}\,\mathbf{k}_\mu-\delta_{j-1}\,\Delta t\,\mathbf{y}'_{i-1}\right)\\
&\quad+[\gamma_{11}(\Delta\tau\,A_{22})^2+(\Delta\tau\,A_{22})]\,\mathbf{z}_{k-1,i}.
\end{aligned}
$$

Proceeding, we have

$$
\begin{aligned}
\widetilde{\mathbf{k}}_3 &= (\Delta\tau\,A_{21})\,\mathbf{y}_{i-1}+(k-1+\gamma_2)(\Delta\tau\,A_{21})\,\Delta\tau\,\mathbf{y}'_{i-1}\\
&\quad+\left(\frac{k-1+\gamma_2}{\delta_{j-1}N}\right)^2(\Delta\tau\,A_{22})\left(\sum_{\mu=1}^{4}\delta_{j-1,\mu}\,\mathbf{k}_\mu-\delta_{j-1}\,\Delta t\,\mathbf{y}'_{i-1}\right)\\
&\quad+\gamma_{21}(\Delta\tau\,A_{22})\,\widetilde{\mathbf{k}}_1+\gamma_{22}(\Delta\tau\,A_{22})\,\widetilde{\mathbf{k}}_2+(\Delta\tau\,A_{22})\,\mathbf{z}_{k-1,i}.
\end{aligned}
$$

Substituting for $\widetilde{\mathbf{k}}_1$ and $\widetilde{\mathbf{k}}_2$, we obtain (for $p=1,2,3$)

$$
\begin{aligned}
\widetilde{\mathbf{k}}_p &= B_p(\Delta\tau\,A_{21})\,\mathbf{y}_{i-1}+C_{pk}(\Delta\tau\,A_{21})\,\Delta\tau\,\mathbf{y}'_{i-1}\\
&\quad+E_{pkj}(\Delta\tau\,A_{21})\left(\sum_{\mu=1}^{4}\delta_{j-1,\mu}\,\mathbf{k}_\mu-\delta_{j-1}\,\Delta t\,\mathbf{y}'_{i-1}\right)+(\Delta\tau\,A_{22})\,B_p\,\mathbf{z}_{k-1,i},
\end{aligned}
$$

where

$$\begin{aligned}
B_1 &= I, \quad B_2 = I + \gamma_{11}(\Delta\tau\, A_{22}),\\
B_3 &= I + (\gamma_{21} + \gamma_{22})(\Delta\tau\, A_{22}) + \gamma_{11}\,\gamma_{22}(\Delta\tau\, A_{22})^2,\\
C_{1k} &= (k-1)\, I, \quad C_{2k} = (k - 1 + \gamma_1)\, I + \gamma_{11}(k-1)(\Delta\tau\, A_{22}),\\
C_{3k} &= (k - 1 + \gamma_2)\, I + \gamma_{21}(k-1)(\Delta\tau\, A_{22})\\
&\quad + \gamma_{22}(k - 1 + \gamma_1)(\Delta\tau\, A_{22}) + \gamma_{11}\,\gamma_{22}(k-1)(\Delta\tau\, A_{22})^2,\\
E_{1kj} &= \left(\frac{k-1}{\delta_{j-1}N}\right)^2 I, \quad E_{2kj} = \left(\frac{k-1+\gamma_1}{\delta_{j-1}N}\right)^2 I + \gamma_{11}\left(\frac{k-1}{\delta_{j-1}N}\right)^2 (\Delta\tau\, A_{22}),\\
E_{3kj} &= \left(\frac{k-1+\gamma_2}{\delta_{j-1}N}\right)^2 I + \gamma_{21}\left(\frac{k-1}{\delta_{j-1}N}\right)^2 (\Delta\tau\, A_{22})\\
&\quad + \gamma_{22}\left(\frac{k-1+\gamma_1}{\delta_{j-1}N}\right)^2 (\Delta\tau\, A_{22}) + \gamma_{11}\,\gamma_{22}\left(\frac{k-1}{\delta_{j-1}N}\right)^2 (\Delta\tau\, A_{22}).
\end{aligned}$$

Substituting $\mathbf{y}'_{i-1} = A_{11}\,\mathbf{y}_{i-1} + A_{12}\,\mathbf{z}_{i-1}$, we get

$$\widetilde{\mathbf{k}}_p = U_{pkj}\,\mathbf{y}_{i-1} + V_{pkj}\,\mathbf{z}_{i-1} + W_{pkj}\sum_{\mu=1}^{4}\delta_{j-1,\mu}\,\mathbf{k}_\mu + K_p\,\mathbf{z}_{k-1,i},$$

where

$$\begin{aligned}
U_{pkj} &= B_p(\Delta\tau\, A_{21}) + C_{pk}(\Delta\tau\, A_{21})(\Delta\tau\, A_{11}) - \delta_{j-1}\,E_{pkj}(\Delta\tau\, A_{21})(\Delta t\, A_{11}),\\
V_{pkj} &= C_{pk}(\Delta\tau\, A_{21})(\Delta\tau\, A_{12}) - \delta_{j-1}\,E_{pkj}(\Delta\tau\, A_{21})(\Delta t\, A_{12}),\\
W_{pkj} &= E_{pkj}(\Delta\tau\, A_{21}), \quad K_p = (\Delta\tau\, A_{22})B_p.
\end{aligned}$$

Now

$$\begin{aligned}
\mathbf{z}_{ki} &= \mathbf{z}_{k-1,i} + \sum_{p=1}^{3}\beta_p\,\widetilde{\mathbf{k}}_p = \left(\sum_{p=1}^{3}\beta_p\,U_{pkj}\right)\mathbf{y}_{i-1} + \left(\sum_{p=1}^{3}\beta_p\,V_{pkj}\right)\mathbf{z}_{i-1}\\
&\quad + \left(\sum_{p=1}^{3}\beta_p\,W_{pkj}\right)\left(\sum_{p=1}^{4}\delta_{j-1,\mu}\,\mathbf{k}_\mu\right) + \widetilde{Y}\,\mathbf{z}_{k-1,i},
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{Y} &= I + \sum_{p=1}^{3}\beta_p\,K_p = I + (\beta_1 + \beta_2 + \beta_3)(\Delta\tau\, A_{22})\\
&\quad + (\beta_2\,\gamma_{11} + \beta_3\,\gamma_{21} + \beta_3\,\gamma_{22})(\Delta\tau\, A_{22})^2 + \beta_3\,\gamma_{11}\,\gamma_{22}(\Delta\tau\, A_{22})^3.
\end{aligned}$$

By induction it can be shown (for $j = 2, 3, 4, 5$) that

$$\mathbf{z}^*\left(t_{i-1} + \delta_{j-1}\,\Delta t,\, \mathbf{y}_{i-1} + \sum_{\mu=1}^{4}\delta_{j-1,\mu}\,\mathbf{k}_\mu\right) = \mathbf{z}_{\delta_{j-1}N,i} = M_j\,\mathbf{y}_{i-1} + N_j\,\mathbf{z}_{i-1} + P_j\left(\sum_{\mu=1}^{4}\delta_{j-1,\mu}\,\mathbf{k}_\mu\right),$$

where

$$\begin{aligned}
M_j &= \sum_{p=1}^{3}\sum_{k=1}^{\delta_{j-1}N}\beta_p\,\widetilde{Y}^{\delta_{j-1}N-k}\,U_{pkj},\\
N_j &= \widetilde{Y}^{\delta_{j-1}N} + \sum_{p=1}^{3}\sum_{k=1}^{\delta_{j-1}N}\beta_p\,\widetilde{Y}^{\delta_{j-1}N-k}\,V_{pkj},\\
P_j &= \sum_{p=1}^{3}\sum_{k=1}^{\delta_{j-1}N}\beta_p\,\widetilde{Y}^{\delta_{j-1}N-k}\,W_{pkj}.
\end{aligned}$$

Now we can continue with the development of $\mathbf{k}_j$ given in equation (5). Thus,

$$\mathbf{k}_1 = (\Delta t\, A_{11})\, \mathbf{y}_{i-1} + (\Delta t\, A_{12})\, \mathbf{z}_{i-1},$$
$$\mathbf{k}_2 = (\Delta t\, A_{11})\, \mathbf{y}_{i-1} + \delta_{11}(\Delta t\, A_{11})\, \mathbf{k}_1 + (\Delta t\, A_{12})\, \mathbf{z}^*(t_{i-1} + \delta_1\, \Delta t,\ \mathbf{y}_{i-1} + \delta_{11}\, \mathbf{k}_1).$$

Substituting for $\mathbf{z}^*$ and then substituting for $\mathbf{k}_1$, we obtain

$$\mathbf{k}_1 = \left[(\Delta t\, A_{11}) + \delta_{11}(\Delta t\, A_{11})^2 + (\Delta t\, A_{12})\, \Lambda_1\right] \mathbf{y}_{i-1} + \Theta_1\, \mathbf{z}_{i-1},$$

where

$$\Lambda_1 = M_2 + \delta_{11}\, P_2(\Delta t\, A_{11}),$$
$$\Theta_1 = (\Delta t\, A_{12})\, N_2 + (\delta_{11}\, \Delta t\, A_{11} + \delta_{11}\, \Delta t\, A_{12}\, P_2)(\Delta t\, A_{12}).$$

Continuing, we have

$$\begin{aligned}\mathbf{k}_3 =& (\Delta t\, A_{11})\, \mathbf{y}_{i-1} + \delta_{21}(\Delta t\, A_{11})\, \mathbf{k}_1 + \delta_{22}(\Delta t\, A_{11})\, \mathbf{k}_2 \\ &+ (\Delta t\, A_{12})\, \mathbf{z}^*(t_{i-1} + \delta_2\, \Delta t,\ \mathbf{y}_{i-1} + \delta_{21}\, \mathbf{k}_1 + \delta_{22}\, \mathbf{k}_2).\end{aligned}$$

Substituting for $\mathbf{z}^*$ and then substituting for $\mathbf{k}_1$ and $\mathbf{k}_2$, we obtain

$$\begin{aligned}\mathbf{k}_3 = \big[&(\Delta t\, A_{11}) + (\delta_{21} + \delta_{22})(\Delta t\, A_{11})^2 + \delta_{11}\, \delta_{22}(\Delta t\, A_{11})^3 \\ &+ (\Delta t\, A_{12})\, \Lambda_2 + \delta_{22}(\Delta t\, A_{11})(\Delta t\, A_{12})\, \Lambda_1\big]\, \mathbf{y}_{i-1} + \Theta_2\, \mathbf{z}_{i-1},\end{aligned}$$

where

$$\Lambda_2 = M_3 + (\delta_{21} + \delta_{22})\, P_3(\Delta t\, A_{11}) + \delta_{11}\, \delta_{22}\, P_3(\Delta t\, A_{11})^2 + \delta_{22}\, P_3(\Delta t\, A_{12})\, \Lambda_1,$$
$$\Theta_2 = (\Delta t\, A_{12})[N_3 + \delta_{21}\, P_3(\Delta t\, A_{12}) + \delta_{22}\, P_3\, \Theta_1] + \delta_{21}(\Delta t\, A_{11})(\Delta t\, A_{12}) + \delta_{22}(\Delta t\, A_{11})\, \Theta_1.$$

Continuing, we have

$$\begin{aligned}\mathbf{k}_4 =& (\Delta t\, A_{11})\, \mathbf{y}_{i-1} + \delta_{31}(\Delta t\, A_{11})\, \mathbf{k}_1 + \delta_{32}(\Delta t\, A_{11})\, \mathbf{k}_2 + \delta_{33}(\Delta t\, A_{11})\, \mathbf{k}_3 \\ &+ (\Delta t\, A_{12})\, \mathbf{z}^*(t_{i-1} + \delta_3\, \Delta t,\ \mathbf{y}_{i-1} + \delta_{31}\, \mathbf{k}_1 + \delta_{32}\, \mathbf{k}_2 + \delta_{33}\, \mathbf{k}_3).\end{aligned}$$

Substituting for $\mathbf{z}^*$ and then for $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$, we obtain

$$\begin{aligned}\mathbf{k}_4 = \big\{&(\Delta t\, A_{11}) + (\delta_{31} + \delta_{32} + \delta_{33})(\Delta t\, A_{11})^2 + (\delta_{11}\, \delta_{32} + \delta_{21}\, \delta_{33} + \delta_{22}\, \delta_{33})(\Delta t\, A_{11})^3 \\ &+ \delta_{11}\, \delta_{22}\, \delta_{33}(\Delta t\, A_{11})^4 + (\Delta t\, A_{12})\, \Lambda_3 + \delta_{32}(\Delta t\, A_{11})(\Delta t\, A_{12})\, \Lambda_1 \\ &+ \delta_{33}(\Delta t\, A_{11})(\Delta t\, A_{12})\, \Lambda_2 + \delta_{22}\, \delta_{33}(\Delta t\, A_{11})^2(\Delta t\, A_{12})\, \Lambda_1\big\}\, \mathbf{y}_{i-1} + \Theta_3\, \mathbf{z}_{i-1},\end{aligned}$$

where

$$\begin{aligned}\Lambda_3 =& M_4 + \delta_{31}\, P_4(\Delta t\, A_{11}) + \delta_{32}\, P_4(\Delta t\, A_{11}) + \delta_{11}\, \delta_{32}\, P_4(\Delta t\, A_{11})^2 \\ &+ \delta_{32}\, P_4(\Delta t\, A_{12})\, \Lambda_1 + \delta_{33}\, P_4(\Delta t\, A_{11}) + \delta_{33}(\delta_{21} + \delta_{22})\, P_4(\Delta t\, A_{11})^2 \\ &+ \delta_{11}\, \delta_{22}\, \delta_{33}\, P_4(\Delta t\, A_{11})^3 + \delta_{33}\, P_4(\Delta t\, A_{12})\, \Lambda_2 + \delta_{22}\, \delta_{33}\, P_4(\Delta t\, A_{11})(\Delta t\, A_{12})\, \Lambda_1, \\ \Theta_3 =& (\Delta t\, A_{12})\, N_4 + \delta_{31}(\Delta t\, A_{11} + \Delta t\, A_{12}\, P_4)(\Delta t\, A_{12}) \\ &+ \delta_{32}(\Delta t\, A_{11} + \Delta t\, A_{12}\, P_4)\, \Theta_1 + \delta_{33}(\Delta t\, A_{11} + \Delta t\, A_{12}\, P_4)\, \Theta_2.\end{aligned}$$

One can see that $\Theta_\mu = (\Delta t\, A_{12})\, N_{\mu+1} + \Theta'_\mu$ where

$$\begin{aligned}&\|\Theta'_\mu\| \le \|\Delta t_s\, A_{12}\|\, \widetilde{\Theta}_\mu, \qquad \text{and where} \\ &\widetilde{\Theta}_1 \le \|\delta_{11}(\Delta t\, A_{11}) + \delta_{11}(\Delta t\, A_{12})\, P_2\|, \\ &\widetilde{\Theta}_2 \le \|\delta_{21}\, P_3(\Delta t\, A_{12})\| + |\delta_{22}|\, \|P_3\|\, \|\Delta t\, A_{12}\| \left(\|N_2\| + \widetilde{\Theta}_1\right) \\ &\qquad + |\delta_{21}|\, \|\Delta t\, A_{11}\| + |\delta_{22}|\, \|\Delta t\, A_{11}\| \left(\|N_2\| + \widetilde{\Theta}_1\right), \\ &\widetilde{\Theta}_3 \le |\delta_{31}|\, \|(\Delta t\, A_{11}) + (\Delta t\, A_{12})\, P_4\| + |\delta_{32}|\, \|(\Delta t\, A_{11}) \\ &\qquad + (\Delta t\, A_{12})\, P_4\| \left(\|N_2\| + \widetilde{\Theta}_1\right) + |\delta_{33}|\, \|(\Delta t\, A_{11}) + (\Delta t\, A_{12})\, P_4\| \left(\|N_3\| + \widetilde{\Theta}_2\right).\end{aligned}$$

Recall that $\mathbf{y}_i = \mathbf{y}_{i-1} + \sum_{q=1}^{4} \alpha_q \mathbf{k}_q$. Substituting for $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$, and $\mathbf{k}_4$, we obtain

$$\sum_{q=1}^{4} \alpha_q \mathbf{k}_q = [(Y - I) + R]\,\mathbf{y}_{i-1} + S\,\mathbf{z}_{i-1},$$

where

$$\begin{aligned}
Y &= I + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)(\Delta t\, A_{11}) + [\alpha_2\, \delta_{11} + \alpha_3(\delta_{21} + \delta_{22}) + \alpha_4(\delta_{31} + \delta_{32} + \delta_{33})](\Delta t\, A_{11})^2 \\
&\quad + [\alpha_3\, \delta_{11}\, \delta_{22} + \alpha_4(\delta_{11}\, \delta_{32} + \delta_{21}\, \delta_{33} + \delta_{22}\, \delta_{33})](\Delta t\, A_{11})^3 + \alpha_4\, \delta_{11}\, \delta_{22}\, \delta_{33}(\Delta t\, A_{11})^4, \\
R &= (\Delta t\, A_{12})(\alpha_2\, \Lambda_1 + \alpha_3\, \Lambda_2 + \alpha_4\, \Lambda_3) + \alpha_3\, \delta_{22}(\Delta t\, A_{11})(\Delta t\, A_{12})\, \Lambda_1 \\
&\quad + \alpha_4\, \delta_{32}(\Delta t\, A_{11})(\Delta t\, A_{12})\, \Lambda_1 + \alpha_4\, \delta_{33}(\Delta t\, A_{11})(\Delta t\, A_{12})\, \Lambda_2 + \alpha_4\, \delta_{22}\, \delta_{33}(\Delta t\, A_{11})^2(\Delta t\, A_{12})\, \Lambda_1, \\
S &= \alpha_1(\Delta t\, A_{12}) + \alpha_2\, \Theta_1 + \alpha_3\, \Theta_2 + \alpha_4\, \Theta_3.
\end{aligned}$$

Then

$$\mathbf{y}_i = (Y + R)\,\mathbf{y}_{i-1} + S\,\mathbf{z}_{i-1},$$

and

$$\begin{aligned}
\mathbf{z}_i &= \mathbf{z}^* \left(t_{i-1} + \Delta t,\ \mathbf{y}_{i-1} + \sum_{q=1}^{4} \alpha_q \mathbf{k}_q \right) \\
&= M_5\, \mathbf{y}_{i-1} + N_5\, \mathbf{z}_{i-1} + P_5 \left(\sum_{q=1}^{4} \alpha_q \mathbf{k}_q \right) \\
&= M_5\, \mathbf{y}_{i-1} + N_5\, \mathbf{z}_{i-1} + P_5[(Y - I + R)\,\mathbf{y}_{i-1} + S\,\mathbf{z}_{i-1}] \\
&= [M_5 + P_5(Y - I + R)]\,\mathbf{y}_{i-1} + (N_5 + P_5 S)\,\mathbf{z}_{i-1}.
\end{aligned}$$

We see that

$$\begin{bmatrix} \mathbf{y}_i \\ \mathbf{z}_i \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} \mathbf{y}_{i-1} \\ \mathbf{z}_{i-1} \end{bmatrix},$$

where

$$\Phi_{11} = Y + R, \quad \Phi_{12} = S, \quad \Phi_{21} = M_5 + P_5(Y - I + R), \quad \Phi_{22} = N_5 + P_5 S.$$

We shall say that the numerical integration procedure, that we have described, will be *absolutely stable* if

$$\lim_{I \to \infty} \left\| \begin{matrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{matrix} \right\|^I = 0,$$

where we will employ max norms.

Assume that there are nonsingular matrices S_1 and S_2 such that $S_1^{-1} A_{11} S_1 = D_1$ and $S_2^{-1} A_{22} S_2 = D_2$, where $D_1 = \mathrm{diag}(\lambda_1, \lambda_2, \dots, , \lambda_m)$ and $D_2 = \mathrm{diag}(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \dots, \widetilde{\lambda}_n)$. Assume that the real parts of all eigenvalues are negative. Now

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} \Phi_{11}^* & \Phi_{12}^* \\ \Phi_{21}^* & \Phi_{22}^* \end{bmatrix} \begin{bmatrix} S_1^{-1} & 0 \\ 0 & S_2^{-1} \end{bmatrix},$$

where Φ_{ij}^* is obtained from the mathematical expression for Φ_{ij} by replacing A_{11}, A_{12}, A_{21}, and A_{22} with D_1, $A_{12}^* = S_1^{-1} A_{12} S_2$, $A_{21}^* = S_2^{-1} A_{21} S_1$, and D_2, respectively; for example,

$$\begin{aligned}
\Phi_{11}^* &= S_1^{-1}\, \Phi_{11}\, S_1 = S_1^{-1}(Y + R)\, S_1 \\
&= S_1^{-1}\, Y S_1 + S_1^{-1}\, R S_1 \\
&= Y^* + (\Delta t\, S_1^{-1}\, A_{12}\, S_2)(\alpha_2\, S_2^{-1}\, \Lambda_1\, S_1 + \alpha_3\, S_2^{-1}\, \Lambda_2\, S_1 + \alpha_4\, S_2^{-1}\, \Lambda_3\, S_1) \\
&\quad + \alpha_3(\delta_{22} + \delta_{32})(\Delta t\, D_1)(\Delta t\, S_1^{-1}\, A_{12}\, S_2)(S_2^{-1}\, \Lambda_1\, S_1) \\
&\quad + \alpha_4\, \delta_{33}(\Delta t\, D_1)(\Delta t\, S_1^{-1}\, A_{12}\, S_2)(S_2^{-1}\, \Lambda_2\, S_1) \\
&\quad + \alpha_4\, \delta_{22}\, \delta_{33}(\Delta t\, D_1)^2(\Delta t\, S_1^{-1}\, A_{12}\, S_2)(S_2^{-1}\, \Lambda_1\, S_1),
\end{aligned}$$

where Y^* is Y with A_{11} replaced by D_1,

$$S_2^{-1}\Lambda_1 S_1 = S_2^{-1} M_2 S_1 + \delta_{11}\Delta t(S_2^{-1} P_2 S_1) D_1,$$

and so on. Similarly, the $*$ superscript will be used on other variables to indicate that A_{11}, A_{22}, A_{12}, and A_{21} have been replaced by D_1, D_2, A_{12}^*, and A_{21}^*.

We see that

$$\left\|\begin{matrix}\Phi_{11} & \Phi_{12}\\ \Phi_{21} & \Phi_{22}\end{matrix}\right\|^I \le \left\|\begin{matrix}S_1 & 0\\ 0 & S_2\end{matrix}\right\| \left\|\begin{matrix}\Phi_{11}^* & \Phi_{12}^*\\ \Phi_{21}^* & \Phi_{22}^*\end{matrix}\right\|^I \left\|\begin{matrix}S_1^{-1} & 0\\ 0 & S_2^{-1}\end{matrix}\right\|.$$

Therefore, our method will be absolutely stable if

$$\left\|\begin{matrix}\Phi_{11}^* & \Phi_{12}^*\\ \Phi_{21}^* & \Phi_{22}^*\end{matrix}\right\| < 1,$$

which will follow if

$$\left.\begin{matrix}\|\Phi_{11}^*\| + \|\Phi_{12}^*\| < 1,\\ \|\Phi_{21}^*\| + \|\Phi_{22}^*\| < 1.\end{matrix}\right\} \tag{6}$$

Inequality (6) will follow if

$$\left.\begin{matrix}\|Y^*\| + \|R^*\| + S^*\| < 1,\\ \|M_5^*\| + \|P_5^*\|\,\|Y^* - I\| + \|P_5^*\|(\|R^*\| + \|S^*\|) + \|N_5^*\| < 1.\end{matrix}\right\} \tag{7}$$

But

$$\|R^*\| + \|S^*\| \le \|\Delta t_S\, A_{12}^*\|\, Z^*,$$

where

$$\begin{aligned} Z^* = &\|\alpha_2\,\Lambda_1^* + \alpha_3\,\Lambda_2^* + \alpha_4\,\Lambda_3^*\| + |\alpha_2\,\delta_{22} + \alpha_4\,\delta_{32}|\,\|\Delta t\, D_1\|\,\|\Lambda_1^*\| \\ &+ |\alpha_4\,\delta_{33}|\,\|\Delta t\, D_1\|\,\|\Lambda_2^*\| + |\alpha_4\,\delta_{22}\,\delta_{33}|\,\|\Delta t\, D_1\|^2\|\Lambda_1^*\| \\ &+ \|\alpha_1\, I + \alpha_2\, N_2^* + \alpha_3\, N_3^* + \alpha_4\, N_4^*\| + |\alpha_2|\,\widetilde{\Theta}_1^* + |\alpha_3|\,\widetilde{\Theta}_2^* + |\alpha_4|\,\widetilde{\Theta}_3^*. \end{aligned}$$

Moreover,

$$\begin{aligned} &\|\alpha_1\, I + \alpha_2\, N_2^* + \alpha_3\, N_3^* + \alpha_4\, N_4^*\| \\ &\quad \le \left\|\alpha_1 I + \alpha_2(\widetilde{Y}^*)^{\delta_1 N} + \alpha_3(\widetilde{Y}^*)^{\delta_2 N} + \alpha_4(\widetilde{Y}^*)^{\delta_3 N}\right\| \\ &\quad\quad + \left\|\sum_{j=2}^{4}\sum_{p=1}^{3}\sum_{k=1}^{\delta_{j-1}N} \alpha_j\,\beta_p(\widetilde{Y}^*)^{\delta_{j-1}N-k}\, V_{pkj}^*\right\| \\ &\quad \le |\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4| + \|\Delta\tau\, D_2\|\left\|\sum_{j=2}^{4}\sum_{p=1}^{3}\alpha_j\,\beta_p\, B_p^*\right\| \\ &\quad\quad + \left\|\sum_{j=2}^{4}\sum_{p=1}^{3}\sum_{k=1}^{\delta_{j-1}N} \alpha_j\,\beta_p(\widetilde{Y}^*)^{\delta_{j-1}N-k}\, V_{pkj}^*\right\|. \end{aligned}$$

Since $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$, we see that $Z^* \le 1 + D^*(\Delta t_S)$ where (thinking of N as fixed) $D^*(\Delta t) \to 0$ as $\Delta t \to 0$.

Inequality (7) will follow if

$$\|Y^*\| + \|\Delta t A_{12}^*\|\, Z^* < 1, \qquad \|M_5^*\| + \|N_5^*\| + \|P_5^*\|\,[\,\|Y^* - I\| + \|\Delta t A_{12}^*\|\, Z^*] < 1. \tag{8}$$

The second inequality of (8) can be written as

$$\left\|\sum_{k=1}^{N}\left(\widetilde{Y}^*\right)^{N-k}\left(\sum_{p=1}^{3}\beta_p\,U^*_{pk5}\right)\right\| + \left\|\left(\widetilde{Y}^*\right)^{N} + \sum_{k=1}^{N}\left(\widetilde{Y}^*\right)^{N-k}\left(\sum_{p=1}^{3}\beta_p\,V^*_{pk5}\right)\right\| \\ + \left\|\sum_{k=1}^{N}\left(\widetilde{Y}^*\right)^{N-k}\left(\sum_{p=1}^{3}\beta_p\,W^*_{pk5}\right)\right\|\omega < 1,$$

where $\omega = \|Y^* - I\| + \|\Delta t\,A^*_{12}\|\,Z^*$. The latter inequality will follow if

$$\left\|\widetilde{Y}^*\right\|^N + \mu\sum_{k=0}^{N-1}\left\|\widetilde{Y}^*\right\|^k < 1, \tag{9}$$

where

$$\begin{aligned}\mu &= \max_k\left(\left\|\sum_{p=1}^{3}\beta_p\,U^*_{pk5}\right\| + \left\|\sum_{p=1}^{3}\beta_p\,V^*_{pk5}\right\| + \omega\left\|\sum_{p=1}^{3}\beta_p\,W^*_{pk5}\right\|\right)\\
&\le \|\Delta\tau\,A^*_{21}\|\left[\left\|\sum_{p=1}^{3}\beta_p\,B^*_p\right\| + \sum_{p=1}^{3}|\beta_p|\Big(\|\Delta\tau\,D_1\|\max_k\|C^*_{pk}\|\right.\\
&\quad\left. + \|\Delta t\,D_1\|\max_k\|E^*_{pk5}\| + \|\Delta\tau\,A^*_{12}\|\,\|C^*_{pk}\| + \|\Delta t\,A_{12}\|\max_k\|E^*_{pk5}\| + \omega\|E^*_{pk5}\|\Big)\right],\\
\mu &\le \|\Delta\tau\,A^*_{21}\|\left[\left|\sum_{p=1}^{3}\beta_p\right| + \widetilde{D}^*(\Delta t)\right],\end{aligned}$$

where

$$\begin{aligned}\widetilde{D}^*(\Delta t) &= |\beta_2\,\gamma_{11} + \beta_3\,\gamma_{21} + \beta_3\,\gamma_{22}|\,\|\Delta\tau\,D_2\|\\
&\quad + |\beta_3\,\gamma_{11}\,\gamma_{22}|\,\|\Delta\tau\,D_2\|^2 + \sum_{p=1}^{3}|\beta_p|\Big(\|\Delta t\,D_1\|\max_k\frac{1}{N}\|C^*_{pk}\|\\
&\quad + \|\Delta t\,D_1\|\max_k\|E^*_{pk5}\| + \|\Delta t\,A^*_{12}\|\max_k\frac{1}{N}\|C^*_{pk}\|\\
&\quad + \|\Delta t\,A^*_{12}\|\max_k\|E^*_{pk5}\| + \omega\max_k\|E^*_{pk5}\|\Big),\end{aligned} \tag{10}$$

and $\widetilde{D}^*(\Delta t) \to 0$ as $\Delta t \to 0$ (thinking of N as fixed). Since $\sum_{p=1}^{3}\beta_p = 1$, we have $\mu \le \|\Delta\tau\,A^*_{21}\|\,\widetilde{Z}^*$ where $\widetilde{Z}^* = 1 + \widetilde{D}^*(\Delta t)$.

In order that (9) will hold, we will require that

$$\left\|\widetilde{Y}^*\right\|^N + \|\Delta\tau\,A^*_{21}\|\,\widetilde{Z}^*\sum_{k=0}^{N-1}\left\|\widetilde{Y}^*\right\|^k < 1;$$

that is,

$$\left\|\widetilde{Y}^*\right\|^N + \|\Delta\tau\,A^*_{21}\|\,\widetilde{Z}^*\,\frac{1-\left\|\widetilde{Y}^*\right\|^N}{1-\left\|\widetilde{Y}^*\right\|} < 1.$$

Since we will insure that $\left\|\widetilde{Y}^*\right\| < 1$, the latter inequality will follow if

$$\left(1-\left\|\widetilde{Y}^*\right\|^N\right)\left(\|\Delta\tau\,A^*_{21}\|\,\widetilde{Z}^* - 1 + \left\|\widetilde{Y}^*\right\|\right) < 0.$$

To insure the latter, we will require that $\left\|\widetilde{Y}^*\right\| + \|\Delta\tau\, A^*_{21}\|\, \widetilde{Z}^* < 1$. Therefore, condition (8) (and hence (6)) will hold if

$$\left.\begin{aligned} \|Y^*\| + \|\Delta t\, A^*_{12}\|\, Z^* < 1, \\ \|Y^*\| + \|\Delta\tau\, A^*_{21}\|\, \widetilde{Z}^* < 1. \end{aligned}\right\} \tag{11}$$

Of course, these conditions imply $\|Y^*\| < 1$ and $\left\|\widetilde{Y}^*\right\| < 1$. Here,

$$\widetilde{Y}^* = I + (\beta_1 + \beta_2 + \beta_3)(\Delta\tau\, D_2) + (\beta_2\gamma_{11} + \beta_3\gamma_{21} + \beta_3\gamma_{22})(\Delta\tau\, D_2)^2 + \beta_3\gamma_{11}\gamma_{22}(\Delta\tau\, D_2)^3.$$

In order that (4) will be a third-order method of numerical integration, we must have

$$\widetilde{Y}^* = I + (\Delta\tau\, D_2) + \frac{1}{2}(\Delta\tau\, D_2)^2 + \frac{1}{6}(\Delta\tau\, D_2)^3.$$

Similarly,

$$\widetilde{Y}^* = I + (\Delta t\, D_1) + \frac{1}{2}(\Delta t\, D_1)^2 + \frac{1}{6}(\Delta t\, D_1)^3 + \frac{1}{24}(\Delta t\, D_1)^4,$$

if (3) is to be a fourth-order method.

We let $w_i = \Delta t\, \lambda_i$ for $i = 1, 2, \ldots, m$ and $\widetilde{w}_j = \Delta\tau\, \widetilde{\lambda}_j$ for $j = 1, 2, \ldots, n$. We let x_i and y_i be the real and imaginary parts of w_i, and we let $\widetilde{x}_j$ and $\widetilde{y}_j$ be the real and imaginary parts of $\widetilde{w}_j$. Then $x_i < 0$ and $\widetilde{x}_j < 0$.

We assume that

$$\begin{aligned} &\|A^*_{12}\|\, Z^* \le b_i |RE(\lambda_i)|, \qquad i = 1, 2, \ldots, m, \\ &\|A^*_{12}\|\, \mathbf{Z} \le \widetilde{b}_j |RE(\widetilde{\lambda}_j)|, \qquad j = 1, 2, \ldots, n, \end{aligned} \tag{12}$$

for some numbers b_i and $\widetilde{b}_j$ such that $b_i < 1$ and $\widetilde{b}_j < 1$. Therefore, (11) will hold provided

$$\begin{aligned} &\left|1 + w_i + \frac{1}{2} w_i^2 + \frac{1}{6} w_i^3 + \frac{1}{24} w_i^4\right| - b_i\, x_i < 1, \qquad i = 1, 2, \ldots, m, \\ &\left|1 + \widetilde{w}_j + \frac{1}{2} \widetilde{w}_j^2 + \frac{1}{6} \widetilde{w}_j^3\right| - \widetilde{b}_j\, \widetilde{x}_j < 1, \qquad j = 1, 2, \ldots, n. \end{aligned} \tag{13}$$

It can be shown that if $b_i \ge 1$ or $\widetilde{b}_j \ge 1$, the corresponding inequality of (13) will have no solution (at least, when $y_i = 0$ or $\widetilde{y}_j = 0$). Each inequality defines a region in the left half-plane of the xy-plane which is tangent to the y-axis at the origin. If $b_i = 0$ or $\widetilde{b}_j = 0$, the corresponding inequality reduces to the classical condition for stability of the fourth-order or third-order Runge-Kutta method. Moreover, smaller values of b_i or $\widetilde{b}_j$ give larger regions of stability.

For the sake of definiteness, we will take $b_i = 1/2$ for $i = 1, 2, \ldots, m$ and $\widetilde{b}_j = 1/2$ for $j = 1, 2, \ldots, n$. Then assumption (12) may be written as

$$\begin{aligned} &\|S_1^{-1} A_{12} S_2\|\, [1 + D^*(\Delta t)] \le \frac{1}{2} \min_i |RE(\lambda_i)|, \\ &\|S_1^{-1} A_{21} S_1\|\, [1 + \widetilde{D}^*(\Delta t)] \le \frac{1}{2} \min_j |RE(\widetilde{\lambda}_j)|. \end{aligned} \tag{14}$$

Thinking of N as fixed, we require that Δt be sufficiently small that (14) will hold. Of course, it must be assumed that

$$\begin{aligned} &\|S_1^{-1} A_{12} S_2\| \le \frac{1}{2} \min_i |RE(\lambda_i)|, \\ &\|S_2^{-1} A_{21} S_1\| \le \frac{1}{2} \min_j |RE(\widetilde{\lambda}_j)|. \end{aligned} \tag{15}$$

We have proven the following theorem.

THEOREM. *Suppose that equations (1) and (2) are integrated with fourth-order and third-order Runge-Kutta methods (3) and (4) using stepsizes Δt and $\Delta\tau = \Delta t/N$, respectively, where $\delta_1 N$,*

$\delta_2 N$, and $\delta_3 N$ are positive integers. Suppose there are non-singular matrices, S_1 and S_2, such that $S_1^{-1} A_{11} S_1 = D_1 = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_m)$ and $S_2^{-1} A_{22} S_2 = D_2 = \mathrm{diag}(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \dots, \widetilde{\lambda}_n)$, where all eigenvalues have negative real parts. Assume that condition (15) holds. Then there are positive functions $D^*(\Delta t)$ and $\widetilde{D}^*(\Delta t)$, such that $\lim_{\Delta t \to 0} D^*(\Delta t) = 0$ and $\lim_{\Delta t \to 0} \widetilde{D}^*(\Delta t) = 0$ and such that the integration of equations (1) and (2) will be absolutely stable for stepsizes Δt and $\Delta\tau$ provided condition (14) holds and condition (13) holds with $b_i = 1/2$, $w_i = \Delta t \lambda_i$, $\widetilde{b}_j = 1/2$, and $\widetilde{w}_j = \Delta\tau\widetilde{\lambda}_j$.

We will now find simplified upper bounds on D^* and $\widetilde{D}^*$, which depend on $\|\Delta t\, D_1\|$, $\|\Delta t\, A_{12}^*\|$, $\|\Delta\tau\, D_2\|$, and $\|\Delta\tau\, A_{21}^*\|$, but which will not depend directly upon N.

Recall that

$$\begin{aligned} D^*(\Delta t) \le{}& \|\alpha_2 \Lambda_1^* + \alpha_3 \Lambda_2^* + \alpha_4 \Lambda_3^*\| + |\alpha_3 \delta_{22} + \alpha_4 \delta_{32}|\, \|\Delta t\, D_1\|\, \|\Lambda_1^*\| + |\alpha_4 \delta_{33}|\, \|\Delta t\, D_1\|\, \|\Lambda_2^*\| \\ &+ |\alpha_4 \delta_{22} \delta_{33}|\, \|\Delta t\, D_1\|^2 \|\Lambda_1^*\| + \|\Delta\tau\, D_2\| \left\| \sum_{j=2}^{4} \sum_{p=1}^{3} \alpha_j \beta_p B_p^* \right\| \\ &+ \left\| \sum_{j=2}^{4} \sum_{p=1}^{3} \sum_{k=1}^{\delta_{j-1}N} \alpha_j \beta_p (\widetilde{Y}^*)^{\delta_{j-1}N-k} V_{pkj}^* \right\| + |\alpha_2|\, \widetilde{\Theta}_1^* + |\alpha_3|\, \widetilde{\Theta}_2^* + |\alpha_4|\, \widetilde{\Theta}_3^*, \end{aligned} \tag{16}$$

where

$$\begin{aligned} &\left\| \sum_{j=2}^{4} \sum_{p=1}^{3} \sum_{k=1}^{\delta_{j-1}N} \alpha_j \beta_p \left(\widetilde{Y}^*\right)^{\delta_{j-1}N-k} V_{pkj}^* \right\| \le \sum_{j=2}^{4} |\alpha_j| \left(\sum_{q=0}^{\delta_{j-1}N-1} \left\|\widetilde{Y}^*\right\|^q \right) \sum_{p=1}^{3} |\beta_p| \max_k \|V_{pkj}^*\| \\ &\quad \le \|\Delta t\, A_{12}^*\| \sum_{j=2}^{4} |\alpha_j| \left(\|\Delta\tau\, A_{21}^*\| \sum_{q=0}^{\delta_{j-1}N-1} \left\|\widetilde{Y}^*\right\|^q \right) \sum_{p=1}^{3} |\beta_p| \max_k \left(\frac{1}{N} \|C_{pk}^*\| + \delta_{j-1} \|E_{pkj}^*\| \right). \end{aligned}$$

But

$$\|\Delta\tau\, A_{21}^*\| \sum_{q=0}^{\delta_{j-1}N-1} \left\|\widetilde{Y}^*\right\|^q = \|\Delta\tau\, A_{21}^*\| \frac{1 - \left\|\widetilde{Y}^*\right\|^{\delta_{j-1}N}}{1 - \left\|\widetilde{Y}^*\right\|},$$

and it is required that $\left\|\widetilde{Y}^*\right\| + \|\Delta\tau\, A_{21}^*\|\, \widetilde{Z}^* < 1$, where $\widetilde{Z}^* \ge 1$. Hence,

$$\|\Delta\tau\, A_{21}^*\| \sum_{q=0}^{\delta_{j-1}N-1} \left\|\widetilde{Y}^*\right\|^q \le 1, \tag{17}$$

so that

$$\begin{aligned} &\left\| \sum_{j=2}^{4} \sum_{p=1}^{3} \sum_{k=1}^{\delta_{j-1}N} \alpha_j \beta_p \left(\widetilde{Y}^*\right)^{\delta_{j-1}N-k} V_{pkj}^* \right\| \\ &\qquad \le \|\Delta t\, A_{12}^*\| \sum_{j=2}^{4} |\alpha_j| \sum_{p=1}^{3} |\beta_p| \max_k \left(\frac{1}{N} \|C_{pk}^*\| + \delta_{j-1} \|E_{pkj}^*\| \right). \end{aligned} \tag{18}$$

Employing (17), one can show that

$$\|M_j^*\| \le \sum_{p=1}^{3} |\beta_p| \left[\|B_p^*\| + \|\Delta t\, D_1\| \max_k \left(\frac{1}{N} \|C_{pk}^*\| + \delta_{j-1} \|E_{pkj}^*\| \right) \right],$$

$$\|N_j^*\| \le 1 + \|\Delta t\, A_{12}^*\| \sum_{p=1}^{3} |\beta_p| \max_k \left(\frac{1}{N} \|C_{pk}^*\| + \delta_{j-1} \|E_{pkj}^*\| \right),$$

$$\|P_j^*\| \le \sum_{p=1}^{3} |\beta_p| \max_k \|E_{pkj}^*\|.$$

Letting

$$\nu_1 = 1, \quad \nu_2 = 1 + \gamma_1 \|\Delta\tau\, D_2\|, \quad \nu_3 = 1 + (|\gamma_{21}| + |\gamma_{22}|)\|\Delta\tau\, D_2\| + |\gamma_{11}\,\gamma_{22}|\,\|\Delta\tau\, D_2\|^2, \tag{19}$$

it can be seen that

$$\|B_p^*\| \le \nu_p, \quad \frac{1}{N}\|C_{pk}^*\| \le \nu_p, \quad \|E_{pkj}^*\| \le \frac{\nu_p}{\delta_{j-1}^2},$$

whence,

$$\begin{aligned}
\|M_j^*\| &\le \left[1 + \|\Delta t\, D_1\| \left(1 + \frac{1}{\delta_{j-1}}\right)\right] \sum_{p=1}^{3} |\beta_p|\, \nu_p, \\
\|N_j^*\| &\le 1 + \|\Delta t\, A_{12}^*\| \left(1 + \frac{1}{\delta_{j-1}}\right) \sum_{p=1}^{3} |\beta_p|\, \nu_p, \\
\|P_j^*\| &\le \frac{1}{\delta_{j-1}^2} \sum_{p=1}^{3} |\beta_p|\, \nu_p.
\end{aligned} \tag{20}$$

Also,

$$\begin{aligned}
\|\Lambda_1^*\| &\le \|M_2^*\| + \delta_1 \|P_2^*\|\, \|\Delta t\, D_1\|, \\
\|\Lambda_2^*\| &\le \|M_3^*\| + \delta_2 \|P_3^*\|\, \|\Delta t\, D_1\| + |\delta_{11}\,\delta_{22}|\, \|P_3^*\|\, \|\Delta t\, D_1\|^2 + |\delta_{22}|\, \|P_3^*\|\, \|\Delta t\, A_{12}^*\|\, \|\Lambda_1^*\|, \\
\|\Lambda_3^*\| &\le \|M_4^*\| + \delta_3 \|P_4^*\|\, \|\Delta t\, D_1\| + |\delta_{11}\,\delta_{32} + \delta_{33}\,\delta_{21} + \delta_{33}\,\delta_{22}|\, \|P_4^*\|\, \|\Delta t\, D_1\|^2 \\
&\quad + |\delta_{32}|\, \|P_4^*\|\, \|\Delta t\, A_{12}^*\|\, \|\Lambda_1^*\| + |\delta_{11}\,\delta_{22}\,\delta_{33}|\, \|P_4^*\|\, \|\Delta t\, D_1\|^3 \\
&\quad + |\delta_{33}|\, \|P_4^*\|\, \|\Delta t\, A_{12}^*\|\, \|\Lambda_2^*\| + |\delta_{22}\,\delta_{33}|\, \|P_4^*\|\, \|\Delta t\, D_1\|\, \|\Delta t\, A_{12}^*\|\, \|\Lambda_1^*\|, \\
\widetilde{\Theta}_1^* &\le \delta_1 \|\Delta t\, D_1\| + \delta_1 \|\Delta t\, A_{12}^*\|\, \|P_2^*\|, \\
\widetilde{\Theta}_2^* &\le |\delta_{21}|\, \|\Delta t\, A_{12}^*\| + \delta_{21} \|\Delta t D_1\| + |\delta_{22}| (\|P_3^*\|\, \|\Delta t\, A_{12}^*\| + \|\Delta t\, D_1\|) \left(\|N_2^*\| + \widetilde{\Theta}_1^*\right), \\
\widetilde{\Theta}_3^* &\le |\delta_{31}| (\|\Delta t\, D_1\| + \|\Delta t\, A_{12}^*\|\, \|P_4^*\|) + |\delta_{32}| (\|\Delta t\, D_1\| + \|\Delta t\, A_{12}^*\|\, \|P_4^*\|) \left(\|N_2^*\| + \widetilde{\Theta}_1^*\right) \\
&\quad + |\delta_{33}| (\|\Delta t\, D_1\| + \|\Delta t\, A_{12}^*\|\, \|P_4^*\|) \left(\|N_3^*\| + \widetilde{\Theta}_2^*\right).
\end{aligned} \tag{21}$$

Recall that $\widetilde{D}^*(\Delta t)$ is given by (10), where

$$\begin{aligned}
\omega \le\ & \|\Delta t\, D_1\| + |\alpha_2\,\delta_1 + \alpha_3\,\delta_2 + \alpha_4\,\delta_3|\, \|\Delta t\, D_1\|^2 \\
& + |\alpha_3\,\delta_{11}\,\delta_{22} + \alpha_4(\delta_{11}\,\delta_{32} + \delta_{21}\,\delta_{33} + \delta_{22}\,\delta_{33})|\, \|\Delta t\, D_1\|^3 \\
& + |\alpha_4\,\delta_{11}\,\delta_{22}\,\delta_{33}|\, \|\Delta t\, D_1\|^4 + \|\Delta t\, A_{12}^*\| [1 + D^*(\Delta t)].
\end{aligned}$$

In summary, (10) and (16) imply

$$D^*(\Delta t) \le \Gamma(\Delta t, \Delta\tau), \quad \widetilde{D}^* \le \widetilde{\Gamma}(\Delta t, \Delta\tau),$$

where

$$\begin{aligned}
\Gamma(\Delta t, \Delta\tau) =\ & |\alpha_2|\, \|\Lambda_1^*\| + |\alpha_3|\, \|\Lambda_2^*\| + |\alpha_4|\, \|\Lambda_3^*\| + |\alpha_3\,\delta_{22} + \alpha_4\,\delta_{32}|\, \|\Delta t\, D_1\|\, \|\Lambda_1^*\| \\
& + |\alpha_4\,\delta_{32}|\, \|\Delta t\, D_1\|\, \|\Lambda_2^*\| + |\alpha_4\,\delta_{22}\,\delta_{33}|\, \|\Delta\, t D_1\|^2 \|\Lambda_1^*\| \\
& + \|\Delta\tau\, D_2\| \left(\sum_{j=2}^{4} |\alpha_j|\right) \sum_{p=1}^{3} |\beta_p|\, \nu_p \\
& + \|\Delta t\, A_{12}^*\| \left[\sum_{j=1}^{4} |\alpha_j| \left(1 + \frac{1}{\delta_{j-1}}\right)\right] \sum_{p=1}^{3} |\beta_p|\, \nu_p + |\alpha_2|\, \widetilde{\Theta}_1^* + |\alpha_3|\, \widetilde{\Theta}_2^* + |\alpha_4|\, \widetilde{\Theta}_3^*,
\end{aligned}$$

$$\widetilde{\Gamma}(\Delta t, \Delta\tau) = |\beta_2\,\gamma_{11} + \beta_3\,\gamma_{21} + \beta_3\,\gamma_{22}|\,\|\Delta\tau\, D_2\| + |\beta_3\,\gamma_{11}\,\gamma_{22}|\,\|\Delta\tau\, D_2|^2$$
$$+ \left(\sum_{p=1}^{3} |\beta_p|\,\nu_p\right)(2\|\Delta t\, D_1\| + 2\|\Delta t\, A_{12}^*\| + \omega),$$
$$\omega \leq \|\Delta t\, D_1\| + |\alpha_3\,\delta_1 + \alpha_2\,\delta_2 + \alpha_4\,\delta_3|\,\|\Delta t\, D_1\|^2$$
$$+ |\alpha_3\,\delta_{11}\,\delta_{22} + \alpha_4(\delta_{11}\,\delta_{32} + \delta_{21}\,\delta_{33} + \delta_{22}\,\delta_{33})|\,\|\Delta t\, D_1\|^3$$
$$+ |\alpha_4\,\delta_{11}\,\delta_{22}\,\delta_{33}|\,\|\Delta t\, D_1\|^4 + \|\Delta t\, A_{12}^*\|\,[1 + \Gamma(\Delta t, \Delta\tau)],$$

with $\|\Lambda_1^*\|$, $\|\Lambda_2^*\|$, $\|\Lambda_3^*\|$, $\widetilde{\Theta}_1^*$, $\widetilde{\Theta}_2^*$, and $\widetilde{\Theta}_3^*$ being given by (21) and with ν_1, ν_2, ν_3, $\|M_2^*\|$, $\|M_3^*\|$, $\|N_2^*\|$, $\|N_3^*\|$, $\|P_2^*\|$, and $\|P_3^*\|$ being given by (19) and (20).

Therefore, the requirement (14) can be replaced by

$$\|A_{12}^*\|\,[1 + \Gamma(\Delta t, \Delta\tau)] \leq \frac{1}{2}\min_i |RE(\lambda_i)|,$$
$$\|A_{21}^*\|\,[1 + \widetilde{\Gamma}(\Delta t, \Delta\tau)] \leq \frac{1}{2}\min_j |RE(\widetilde{\lambda}_j)|.$$

We observe that, as $\Delta t \to 0$ and $\Delta\tau \to 0$, $\widetilde{\Gamma} \to 0$ but

$$\Gamma \to \left(\sum_{p=1}^{3} |\beta_p|\right)\left(\sum_{q=2}^{4} |\alpha_q|\right).$$

(The use of (17), for simplifying the conditions, has prevented Γ from approaching zero.)

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